

Symmetric (not Complete Intersection) Semigroups Generated by Five Elements

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Abstract

We consider symmetric (not complete intersection) numerical semigroups S_5 , generated by five elements, and derive inequalities for degrees of syzygies of S_5 and find the lower bound F_{5s} for their Frobenius numbers. We study a special case W_5 of such semigroups, which satisfy the Watanabe Lemma [14], and show that the lower bound F_{5w} for W_5 is stronger than F_{5s} , $F_{5w} > F_{5s}$.

Keywords: symmetric (not complete intersection) semigroups, Frobenius number

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1 Introduction

Let a numerical semigroup $S_m = \langle d_1, \dots, d_m \rangle$ be generated by a set of m integers $\{d_1, \dots, d_m\}$ such that $\gcd(d_1, \dots, d_m) = 1$, where d_1 and m denote multiplicity and embedding dimension (*edim*) of S_m . There exist $m-1$ polynomial identities [6] for degrees of syzygies associated with semigroup ring $k[S_m]$. They are a source of various relations for semigroups of different nature, e.g., the lower bound F_{CI_m} for the Frobenius number F and degrees e_j of the 1st syzygy of complete intersection (CI) semigroups read

$$F_{CI_m} = (m-1)^{m-1} \sqrt[m]{\pi_m} - \sigma_1, \quad \prod_{j=1}^{m-1} e_j = \pi_m, \quad \pi_m = \prod_{j=1}^m d_j, \quad \sigma_1 = \sum_{j=1}^m d_j,$$

(see [6], Corollaries 1 and 2). The next nontrivial case is a symmetric (not CI) semigroup generated by $m \geq 4$ integers. In [8] such semigroups with $m = 4$ were studied and the lower bound for F was found. In the present paper we deal with more difficult case of symmetric (not CI) semigroups with $m = 5$.

Consider a symmetric numerical semigroup S_5 , which is not CI, and generated by five elements d_i arranged in an ascendant order, $d_j < d_{j+1}$. Its Hilbert series $H(S_5; t)$ with Betti's numbers β_1, β_2 reads,

$$H(S_5; t) = \frac{Q_5(t)}{\prod_{i=1}^5 (1 - t^{d_i})}, \quad Q_5(t) = 1 - \sum_{j=1}^{\beta_1} t^{x_j} + \sum_{j=1}^{\beta_2} t^{y_j} - \sum_{j=1}^{\beta_1} t^{z_j} + t^g, \quad (1)$$

$$z_j = g - x_j, \quad y_{\beta_2-j+1} = g - y_j, \quad g > x_j, y_j, z_j, \quad x_j, y_j, z_j, g \in \mathbb{N}, \quad \beta_2 = 2(\beta_1 - 1).$$

and the Frobenius number is defined as follows, $F(S_5) = g - \sigma_1$. There are two constraints more, $\beta_1 > 4$ and $d_1 > 5$. The inequality $\beta_1 > 4$ holds since a semigroup S_5 is not CI, and the condition $d_1 > 5$ is necessary since the numerical semigroup $\langle m, d_2, \dots, d_m \rangle$ is never symmetric [7].

Let $A = k[p_1, \dots, p_5]$ be a polynomial ring over a field k and $S_5 = \langle d_1, \dots, d_5 \rangle$, then $k[S_5] = k[t^{d_1}, \dots, t^{d_5}]$ is the semigroup ring of S_5 . Let a map $\phi : A \rightarrow k[t]$ be defined by $p_i \mapsto t^{d_i}$. If we let $\deg p_i = d_i$, then this map is homogeneous of degree 0. If $R = A/I$ is any graded k -algebra and I is a graded ideal in A , then a minimal graded free resolution F_I of R over A reads,

$$F_I : \quad 0 \longrightarrow A \xrightarrow{\phi_4} A^{\beta_1} \xrightarrow{\phi_3} A^{\beta_2} \xrightarrow{\phi_2} A^{\beta_1} \xrightarrow{\phi_1} A \longrightarrow 0. \quad (2)$$

In (2) the maps ϕ_1 and ϕ_4 comprise the binomial generators with row- and column-vectors representations, respectively. The map ϕ_1 is of special interest,

$$\begin{aligned} \phi_1 = (f_1, \dots, f_{\beta_1}), \quad f_i = \prod_{j=1}^5 p_j^{a_{ij}} - \prod_{j=1}^5 p_j^{h_{ij}}, \quad a_{ij}, h_{ij} \in \mathbb{N} \cup \{0\}, \\ a_{i1} + \dots + a_{i5} \geq 2, \quad h_{i1} + \dots + h_{i5} \geq 2, \quad \forall \quad 1 \leq i \leq \beta_1. \end{aligned} \quad (3)$$

The other two maps ϕ_2 and ϕ_3 comprise the monomial generators with $(\beta_1 \times \beta_2)$ and $(\beta_2 \times \beta_1)$ matrix representations, respectively.

Bearing in mind a map $p_i \mapsto t^{d_i}$, write a minimal relation

$$\sum_{j=1}^5 a_{ij} d_j = \sum_{j=1}^5 h_{ij} d_j, \quad (4)$$

corresponded to f_i in (3), and assume that all cancellations of similar terms in (4) are performed, i.e., no common elements d_j appear in its r.h.s. and l.h.s., e.g., $a_{13}d_3 + a_{15}d_5 = h_{11}d_1 + h_{12}d_2 + h_{14}d_4$.

For convenience, let us choose matrices a_{ij}, h_{ij} in such a way that the smallest element d_1 appears only in the r.h.s. of (4), i.e., the summation index j in the l.h.s. of (4) is always running in the range $2 \leq j \leq 5$. This reduction allows to make use of a *staircase diagram*¹ for a generic monomial ideal in the 4-dim integer lattice \mathbb{Z}^4 (see [12], sections 3, 6). The degrees x_j of the 1st syzygy of S_5 are built as linear combinations of elements of the set $\{d_2, d_3, d_4, d_5\}$, i.e.,

$$x_i = \sum_{j=2}^5 a_{ij} d_j, \quad a_{ij} \in \mathbb{N} \cup \{0\}, \quad a_{i2} + \dots + a_{i5} \geq 2, \quad \forall \quad 1 \leq i \leq \beta_1. \quad (5)$$

In fact, the values of x_i , as well as the other syzygies degrees y_i, z_i and g , do not depend on the chosen 4-dim bases, $\{d_2, d_3, d_4, d_5\}$, $\{d_1, d_3, d_4, d_5\}$, $\{d_1, d_2, d_4, d_5\}$, $\{d_1, d_2, d_3, d_5\}$ or $\{d_1, d_2, d_3, d_4\}$. The only difference appears in the shapes of the five staircase diagrams related to the corresponding bases.

¹For the first time the staircase diagrams are introduced for 3- and 4-generated semigroups in [2] and [10], respectively.

Among β_1 degrees x_i there exist five entries such that $x_j = v_{jj}d_j$ and neither integer $v < v_{jj}$ gives rise to the degree vd_j which is contained in the set $\{x_j\}$ in the sense of (5), i.e.,

$$v_{jj} = \min \left\{ v_{jj} \mid v_{jj} \geq 2, v_{jj}d_j = \sum_{i=2, i \neq j}^5 v_{ji}d_i, v_{ji} \in \mathbb{N} \cup \{0\} \right\}, \quad 1 \leq j \leq 5, \quad v_{jj} \in \{a_{ij}\}. \quad (6)$$

The degrees y_i and z_i of the 2nd and 3rd syzygies read [12],

$$y_i = \sum_{j=1}^{\beta_1} b_{ij}x_j + \sum_{j=2}^5 c_{ij}d_j, \quad z_i = \sum_{j=1}^{\beta_2} l_{ij}y_j + \sum_{j=2}^5 e_{ij}d_j, \quad b_{ij}, l_{ij} \in \{0, 1\}, \quad c_{ij}, e_{ij} \in \mathbb{N} \cup \{0\}, \quad (7)$$

where

$$\text{for every } j, \quad 1 \leq j \leq \beta_1, \quad \exists \quad \text{at least one } i, \quad 1 \leq i \leq \beta_2, \quad \text{such that } b_{ij} = 1, \quad (8)$$

$$\text{for every } j, \quad 1 \leq j \leq \beta_2, \quad \exists \quad \text{at least one } i, \quad 1 \leq i \leq \beta_1, \quad \text{such that } l_{ij} = 1, \quad (9)$$

$$\text{for every } j, \quad 2 \leq j \leq 5, \quad \exists \quad \text{at least one } i, \quad 1 \leq i \leq \beta_2, \quad \text{such that } c_{ij} \geq 1, \quad (10)$$

$$\text{for every } j, \quad 2 \leq j \leq 5, \quad \exists \quad \text{at least one } i, \quad 1 \leq i \leq \beta_1, \quad \text{such that } e_{ij} \geq 1. \quad (11)$$

Denote the k -th power symmetric polynomials

$$X_k(x_1, \dots, x_{\beta_1}) = \sum_{j=1}^{\beta_1} x_j^k, \quad Y_k(y_1, \dots, y_{\beta_2}) = \sum_{j=1}^{\beta_2} y_j^k, \quad Z_k(z_1, \dots, z_{\beta_1}) = \sum_{j=1}^{\beta_1} z_j^k, \quad (12)$$

for short by X_k, Y_k, Z_k and prove an auxiliary Lemma.

Lemma 1 *Let a symmetric (not CI) numerical semigroup S_5 be given with its Hilbert series $H(S_5; z)$ according to (1). Then the following inequalities hold,*

$$X_1 \leq \frac{\beta_1 g}{2} - (\sigma_1 - d_1), \quad Z_1 \geq \frac{\beta_1 g}{2} + (\sigma_1 - d_1). \quad (13)$$

Proof Combine (5) and (7) and find a sum $Z_1 = \sum_{j=1}^{\beta_1} z_j$,

$$Z_1 = \sum_{i=1}^{\beta_1} B_i x_i + \sum_{i=2}^5 D_i d_i, \quad B_i = \sum_{j=1}^{\beta_1} \sum_{k=1}^{\beta_2} l_{jk} b_{ki}, \quad D_i = \sum_{j=1}^{\beta_1} \sum_{k=1}^{\beta_2} l_{jk} c_{ki} + \sum_{k=1}^{\beta_1} e_{ki}. \quad (14)$$

Prove that $B_i \geq 1$. Indeed, according to (8), for given i there exists at least one k^* such that $b_{k^*i} = 1$ and, by virtue of (14), we get $B_i \geq \sum_{j=1}^{\beta_1} l_{jk^*}$. However, according to (9), for given k^* there exists at least one j^* such that $l_{j^*k^*} = 1$ and finally, $B_i \geq 1$.

Proof of another inequality, $D_i \geq 2$, is similar. Indeed, according to (10), for given i there exist at least one k^* and one k^{**} such that $c_{k^*i}, e_{k^{**}i} \geq 1$. Then, by (14), we get $D_i \geq \sum_{j=1}^{\beta_1} l_{jk^*} + 1 \geq 2$. Making use of these inequalities, $B_i \geq 1$ and $D_i \geq 2$, and comparing Z_1 and X_1 , we conclude,

$$Z_1 \geq X_1 + 2(\sigma_1 - d_1). \quad (15)$$

On the other hand, performing a summation over j in relation $z_j = g - x_j$, given in (1), we get, $Z_1 + X_1 = \beta_1 g$. Combining the last equality with (15) we arrive at (13). \square

2 Identities for degrees of syzygies for symmetric (not CI) semigroup S_5

Write four polynomial identities derived in [6], formulas (6.12),

$$\begin{aligned} Y_1 &= \chi g, & Y_1(2X_1 - Y_1) + \chi(Y_2 - 2X_2) &= 0, & Y_1^2(3X_1 - Y_1) - 3\chi X_2 Y_1 + \chi^2 Y_3 &= 0, \\ Y_1^3(4X_1 - Y_1) - 6\chi X_2 Y_1^2 + 4\chi^2 X_3 Y_1 + \chi^3(Y_4 - 2X_4) &= 24\chi^3 \pi_5, & \chi &= \beta_1 - 1. \end{aligned} \quad (16)$$

Represent equalities (16) in more convenient form

$$\begin{aligned} \chi g^2 - 2X_1 g + 2X_2 - Y_2 &= 0, & \chi g^3 - 3X_1 g^2 + 3X_2 g - Y_3 &= 0, \\ \chi g^4 - 4X_1 g^3 + 6X_2 g^2 - 4X_3 g + 2X_4 - Y_4 + 24\pi_5 &= 0. \end{aligned} \quad (17)$$

Two first equalities in (17) are consistent if

$$\chi g^3 - 3Y_2 g + 2Y_3 = 0. \quad (18)$$

Consider elementary symmetric polynomials $W_k = W_k(w_1, \dots, w_n)$ in n positive variables w_j

$$W_0 = 1, \quad W_1 = \sum_{j=1}^n w_j, \quad W_2 = \sum_{i \geq j=1}^n w_i w_j, \quad W_3 = \sum_{i \geq j \geq r=1}^n w_i w_j w_r, \quad \dots, \quad W_n = \prod_{j=1}^n w_j, \quad (19)$$

which are related to each other by the Newton-Maclaurin inequalities [11],

$$\frac{W_1}{n} \geq \left(\frac{W_2}{\binom{n}{2}} \right)^{\frac{1}{2}} \geq \left(\frac{W_3}{\binom{n}{3}} \right)^{\frac{1}{3}} \geq \dots \geq W_n^{\frac{1}{n}}, \quad \left(\frac{W_r}{\binom{n}{r}} \right)^2 \geq \frac{W_{r-1}}{\binom{n}{r-1}} \frac{W_{r+1}}{\binom{n}{r+1}}. \quad (20)$$

Equalities in (20) are attainable iff $w_1 = \dots = w_n$.

Define two other elementary symmetric polynomials in variables $\{x_j\}$ and $\{y_j\}$, respectively,

$$U_k = W_k(x_1, \dots, x_{\beta_1}), \quad V_k = W_k(y_1, \dots, y_{\beta_2}). \quad (21)$$

Both sequences U_1, \dots, U_{β_1} and V_1, \dots, V_{β_2} satisfy inequalities (20), e.g.,

$$U_r \leq \binom{\beta_1}{r} \left(\frac{U_1}{\beta_1} \right)^r, \quad V_r \leq \binom{\beta_2}{r} \left(\frac{V_1}{\beta_2} \right)^r. \quad (22)$$

In sequel we make use of additional inequalities which can be derived combining different parts of (20),

$$W_1 W_r - (r+1) W_{r+1} \geq \frac{r}{\beta_1} W_1 W_r, \quad 1 \leq r < n. \quad (23)$$

Recall the Newton recursion identities for symmetric polynomials X_r and U_r ,

$$\begin{aligned} X_1 &= U_1, & X_2 &= U_1 X_1 - 2U_2, & X_3 &= U_1 X_2 - U_2 X_1 + 3U_3, \\ X_4 &= U_1 X_3 - U_2 X_2 + U_3 X_1 - 4U_4, & \dots, \end{aligned}$$

and write explicit expressions for the four first of them, $r = 1, 2, 3, 4$,

$$\begin{aligned} X_1 &= U_1, & X_2 &= U_1^2 - 2U_2, & X_3 &= U_1^3 - 3U_2U_1 + 3U_3, \\ X_4 &= U_1^4 + 2U_2^2 - 4U_1^2U_2 + 4U_1U_3 - 4U_4. \end{aligned} \quad (24)$$

The similar relations may be written for another pair Y_r and V_r ,

$$\begin{aligned} Y_1 &= V_1, & Y_2 &= V_1^2 - 2V_2, & Y_3 &= V_1^3 - 3V_2V_1 + 3V_3, \\ Y_4 &= V_1^4 + 2V_2^2 - 4V_1^2V_2 + 4V_1V_3 - 4V_4. \end{aligned} \quad (25)$$

Substitute (24, 25) into (18) and into the 1st and 3rd equalities of (17) and obtain,

$$V_2 = \frac{\chi(\chi-1)}{2}g^2 + U_1g - U_1^2 + 2U_2, \quad (26)$$

$$V_3 = (\chi-1) \left(V_2 - \frac{\chi(2\chi-1)}{6}g^2 \right) g, \quad (27)$$

$$\begin{aligned} V_4 &= \frac{\chi(\chi^3-1)}{4}g^4 + U_1g^3 - \left(\frac{3U_1^2}{2} - 3U_2 + \chi^2V_2 \right) g^2 + \\ &\quad (U_1^3 - 3U_1U_2 + 3U_3 + \chi V_3) g - \left(\frac{U_1^4}{2} + U_2^2 - 2U_1^2U_2 + 2U_1U_3 - 2U_4 - \frac{V_2^2}{2} + 6\pi_5 \right). \end{aligned} \quad (28)$$

Thus, we have

$$V_2 = K_2g^2 + K_1g - K_0, \quad (29)$$

$$K_2 = \binom{\chi}{2}, \quad K_1 = U_1, \quad K_0 = U_1^2 - 2U_2,$$

$$V_3 = (L_2g^2 + L_1g - L_0)g, \quad (30)$$

$$L_2 = \binom{\chi}{3}, \quad L_1 = (\chi-1)U_1, \quad L_0 = (\chi-1)(U_1^2 - 2U_2),$$

$$V_4 = M_4g^4 + M_3g^3 - M_2g^2 - M_1g + M_0 - 6\pi_5, \quad (31)$$

$$M_4 = \binom{\chi}{4}, \quad M_3 = \binom{\chi-1}{2}U_1, \quad M_2 = \binom{\chi-1}{2}(U_1^2 - 2U_2) - U_2,$$

$$M_1 = U_1U_2 - 3U_3, \quad M_0 = U_2^2 - 2U_1U_3 + 2U_4.$$

In (29 – 31) all coefficients K_0, L_0, M_0, M_1, M_2 are positive in accordance with (22,23),

$$K_0 = \frac{L_0}{\chi-1} \geq \frac{U_1^2}{\beta_1},$$

$$M_0 = \sum_{i \geq j=1}^{\beta_1} x_i^2 x_j^2, \quad M_1 \geq 2 \frac{U_1U_2}{\beta_1},$$

$$M_2 \geq \frac{\beta_1^2 - 6\beta_1 + 7}{2\beta_1} U_1^2 + \binom{\beta_1}{2} \left(\frac{U_1^2}{\beta_1^2} - \frac{U_2}{\binom{\beta_1}{2}} \right) > 0 \quad \text{if } \beta_1 > 4.$$

3 Inequalities for symmetric polynomials U_r and V_r

Combine equalities (29,30) with two inequalities (22) for V_2, V_3 ,

$$a) \quad K_2 g^2 + K_1 g - K_0 \leq \binom{2\chi}{2} \left(\frac{V_1}{2\chi} \right)^2, \quad b) \quad L_2 g^2 + L_1 g - L_0 \leq \binom{2\chi}{3} \left(\frac{V_1}{2\chi} \right)^3 \frac{1}{g}, \quad (32)$$

where χ is defined in (16). Substituting K_j, L_j into (32) we conclude that both inequalities (32a) and (32b) result in a single common inequality. Indeed, after simple calculations we obtain,

$$\begin{aligned} a) \quad U_1 g + 2U_2 - U_1^2 &\leq g^2 \left[\frac{1}{4} \binom{2\chi}{2} - \binom{\chi}{2} \right] = \frac{\chi}{4} g^2, \\ b) \quad U_1 g + 2U_2 - U_1^2 &\leq \frac{g^2}{\chi - 1} \left[\frac{1}{8} \binom{2\chi}{3} - \binom{\chi}{3} \right] = \frac{\chi}{4} g^2. \end{aligned} \quad (33)$$

Introduce a new variable $u_n = U_n/g^n$ and prove Lemma.

Lemma 2 *Let a symmetric (not CI) numerical semigroup S_5 be given with its Hilbert series $H(S_5; z)$ according to (1). Then the following inequalities hold:*

$$\text{if } 0 < u_1 \leq u_1^-, \quad \text{then } 0 < u_2 \leq \Psi_1(u_1), \quad \Psi_1(u_1) = \frac{\chi}{2\beta_1} u_1^2, \quad (34)$$

$$\text{if } u_1^- \leq u_1 \leq u_\odot, \quad \text{then } 0 < u_2 \leq \Psi_2(u_1), \quad \Psi_2(u_1) = \frac{1}{2} \left(u_1^2 - u_1 + \frac{\chi}{4} \right), \quad (35)$$

where $u_1^- = (\beta_1 - \sqrt{\beta_1})/2$, $u_2^* = \Psi_1(u_1^-) = \Psi_2(u_1^-)$, $u_\odot = \beta_1/2 - \varepsilon$, $\varepsilon = (\sigma_1 - d_1)/g$.

Proof Write inequality (33) as follows,

$$u_1^2 - 2u_2 \geq u_1 - \chi/4. \quad (36)$$

On the other hand, by (23) the integers u_1, u_2 satisfy

$$u_1^2 - 2u_2 \geq u_1^2/\beta_1. \quad (37)$$

Inequality (37) holds always while inequality (36) is valid not for every u_1 . In order to make both inequalities consistent we have to find such range of u_1 where both inequalities (36,37) are satisfied for any u_1 within the range. First, consider an inequality between the r.h.s. in (36,37).

$$u_1 - \chi/4 \geq u_1^2/\beta_1. \quad (38)$$

Solving the above quadratic inequality, we obtain

$$u_1^- \leq u_1 \leq u_1^+, \quad u_1^\pm = \frac{\beta_1 \pm \sqrt{\beta_1}}{2}. \quad (39)$$

However, by Lemma 1 there exists more strong upper bound $u_1 \leq u_\odot < u_1^+$, where $u_\odot = \beta_1/2 - (\sigma_1 - d_1)/g$. Thus, combining both inequalities we arrive at (35).

In the opposite case, $0 < u_1 \leq u_1^-$, both inequalities (36,37) are still holding, but a sign in (38) has to be inverted and we arrive at (34). \square

Apply a similar arguments to equality (31) and get,

$$M_3g^3 - M_2g^2 - M_1g + M_0 \leq \left[\frac{1}{16} \binom{2\chi}{4} - \binom{\chi}{4} \right] g^4 + 6\pi_5.$$

Substituting M_j from (31) into the last inequality and simplifying it, we arrive at

$$N_2 + N_1 + N_0 \leq J(g), \quad J(g) = \frac{\chi(\chi-1)(4\chi-7)}{32} + \frac{6\pi_5}{g^4}, \quad (40)$$

$$N_2 = \binom{\chi-1}{2}(u_1 - u_1^2 + 2u_2), \quad N_1 = u_2 - u_1u_2 + 3u_3, \quad N_0 = u_2^2 - 2u_1u_3 + 2u_4. \quad (41)$$

Denote by $N = N_2 + N_1 + N'_0$ and rewrite inequality (40) as follows

$$N \leq J(g) + \frac{3(\beta_1+1)}{2\beta_1}u_1u_3, \quad N'_0 = N_0 + \frac{3(\beta_1+1)}{2\beta_1}u_1u_3.$$

Make use of the 2nd inequality in (20) with $r = 2$, i.e., $3(\beta_1-1)u_1u_3 \leq 2(\beta_1-2)u_2^2$, and obtain

$$N \leq J(g) + \frac{(\beta_1+1)(\beta_1-2)}{\beta_1(\beta_1-1)}u_2^2. \quad (42)$$

On the other hand, according to the 1st inequality in (20), $r = 2, 3$, we get

$$N_1 \leq u_2 \left(1 - \frac{2u_1}{\beta_1} \right), \quad N'_0 \leq u_2^2, \quad (43)$$

so that another inequality holds,

$$N \leq \binom{\chi-1}{2}(u_1 - u_1^2 + 2u_2) + u_2 \left(1 - \frac{2u_1}{\beta_1} \right) + u_2^2. \quad (44)$$

Inequalities (44) hold always while inequality (42) is valid not for every u_1, u_2 . To make both inequalities consistent we have to find such ranges of u_1, u_2 where both inequalities (42,44) are satisfied for any u_1, u_2 within these ranges. In order to provide this statement we have to require that the upper bound of N in (42) exceeds the upper bound in (44),

$$\binom{\beta_1-2}{2}(u_1 - u_1^2 + 2u_2) + u_2 \left(1 - \frac{2u_1}{\beta_1} \right) + u_2^2 \leq J(g) + \frac{(\beta_1+1)(\beta_1-2)}{\beta_1(\beta_1-1)}u_2^2,$$

where $J(g)$ is defined in (40). Otherwise some solutions of (44) may not satisfy (42). Simplifying the above inequality in u_1, u_2 we arrive at following representation,

$$\Phi(u_1, u_2) \leq J(g), \quad \Phi(u_1, u_2) = Au_2^2 - 2Bu_1u_2 - Cu_1^2 + Du_2 + Cu_1, \quad (45)$$

$$A = \binom{\beta_1}{2}^{-1}, \quad B = \frac{1}{\beta_1}, \quad C = \binom{\beta_1-2}{2}, \quad D = \beta_1^2 - 5\beta_1 + 7.$$

4 Two lemmas on the greatest lower bound of g

A brief analysis of the master inequality (45) with $J(g)$, given in (40), shows that small values of g in denominator in $J(g)$ will definitely provide the validity of (45), since, due to Lemma 2, both variables u_1, u_2 are running in the finite ranges. Our goal in this section is to find *the greatest lower bound* $GLB_g = g_s$ which is dependent on generators d_j and Betti number β_1 only and provides the validity of inequality (45) for all u_1, u_2 given in (34, 35).

The name *greatest* should not mislead the readers, since in its rigorous sense (not related to inequality (45)) for every numerical semigroup S_5 the GLB_g is equal g , which appears in the Hilbert series (1) and is larger than any reasonable estimate g_s , however is indeterminable analytically. Different approaches to the GLB_g estimation problem in numerical semigroups may reach different values of g_s . Sometimes it leads to improvement of early results.² But, an enhancement of estimates may occur also for a subset $\{W_5\}$ of symmetric (not CI) semigroups S_5 if semigroups W_5 have possessed an additional property; then GLB_g for W_5 will be stronger than for the rest of S_5 . That is exactly the case of symmetric (not CI) semigroups S_5 satisfied the Watanabe Lemma [14] and discussed in section 5.

Here we prove two Lemmas 3, 4 on GLB_g , related to the master inequality (45), in two different cases separately, when the variable u_1 is running in the ranges $0 < u_1 \leq u_1^-$ and $u_1^- \leq u_1 \leq u_\odot$.

Lemma 3 *Let a symmetric (not CI) numerical semigroup S_5 be given with its Hilbert series $H(S_5; z)$ according to (1). If $u_1 \in [0, u_1^-]$ then the following inequality holds,*

$$g \geq g_s, \quad g_s = 4\lambda(\beta_1)\sqrt[4]{\pi_5}, \quad \lambda(\beta_1) = \sqrt[4]{\frac{3}{4} \frac{\beta_1}{\beta_1 - 1}}, \quad \lambda(\beta_1) < 1. \quad (46)$$

Proof Consider an inequality (45) where u_1 is running in the range $[0, u_1^-]$,

$$Au_2^2 + Du_2 \leq J(g) + 2Bu_1u_2 + Cu_1(u_1 - 1) \leq J(g) + 2Bu_1^-u_2 + Cu_1^-(u_1^- - 1),$$

and rewrite it as follows,

$$Au_2^2 + E_*u_2 - [J(g) + Cu_1^-(u_1^- - 1)] \leq 0, \quad E_* = D - 2Bu_1^- = (\beta_1 - 2)(\beta_1 - 3) + \frac{1}{\sqrt{\beta_1}}, \quad (47)$$

The last inequality results in

$$u_2 \leq \Psi_3(g), \quad \Psi_3(g) = \frac{1}{2A} \left(\sqrt{E_*^2 + 4A [J(g) + Cu_1^-(u_1^- - 1)]} - E_* \right). \quad (48)$$

Combining inequality in (48) with another inequality (34) for u_2 , which holds always, we conclude: if $\Psi_3(g) \leq u_2^*$ then $u_2 \leq u_2^*$. Insert expression (48) for $\Psi_3(g)$ into inequality $\Psi_3(g) \leq u_2^*$ and obtain,

$$A(u_2^*)^2 + E_*u_2^* \geq J(g) + Cu_1^-(u_1^- - 1). \quad (49)$$

²The GLB_g for nonsymmetric numerical semigroup $\langle d_1, d_2, d_3 \rangle$ was calculated in [13, 4], $g_{3*} = \sqrt{3}\sqrt{d_1d_2d_3}$ and improved slightly in [5], $g_{3*} = \sqrt{3}\sqrt{d_1d_2d_3 + 1}$.

Substitute into inequality (49) expressions for u_1^- , A and E_* from (34,45,47) and after detailed algebra and careful inspection we obtain finally,

$$(\beta_1 - 1)^2 + 4\beta_1(\beta_1 - 2)(\beta_1 - 3) - \beta_1(\beta_1 - 2)(4\beta_1 - 11) \geq \frac{192\pi_5\beta_1}{g^4(\beta_1 - 1)}.$$

Simplifying the last inequality we arrive at (46) with $\lambda(\beta_1) < 1$ since $\beta_1 > 4$. \square

In order to find GLB_g in the range $u_1 \in [u_1^-, u_\odot]$ consider two one-parametric roots $\Psi_4^-(u_1, g)$ and $\Psi_4^+(u_1, g)$ of quadratic in u_2 equation $\Phi(u_1, u_2) = J(g)$,

$$\Psi_4^\pm(u_1, g) = \frac{1}{2A} \left(\pm \sqrt{E^2(u_1) + 4A[J(g) + Cu_1(u_1 - 1)]} - E(u_1) \right), \quad E(u_1) = D - 2Bu_1, \quad (50)$$

so that $\Psi_3(g) = \Psi_4^+(u_1^-, g)$. The function $\Psi_4^+(u_1, g)$ plays a key role in determination of GLB_g .

Proposition 1 *In the domain $\mathbb{D} = \{u_1^- < u_1 < u_\odot; u_2 > 0\}$ the following inequalities are equivalent:*

$$a) \quad \Phi(u_1, u_2) \leq J(g) \quad \Longleftrightarrow \quad b) \quad 0 < u_2 \leq \Psi_4^+(u_1, g). \quad (51)$$

Proof $a) \Rightarrow b)$. Since $u_1 > u_1^- > 1$ and $A[J(g) + Cu_1(u_1 - 1)] > 0$ then two roots of quadratic in u_2 equation $\Phi(u_1, u_2) = J(g)$ are of different signs. Namely, $\Psi_4^-(u_1, g) < 0$ and $\Psi_4^+(u_1, g) > 0$. Then, the first inequality in (51a) leads to (51b) in domain \mathbb{D} .

$b) \Rightarrow a)$. Substituting $(u_1, u_2) \in \{u_1^- < u_1 < u_\odot; 0 < u_2 \leq \Psi_4^+(u_1, g)\}$ into $\Phi(u_1, u_2)$ in (45) we arrive at (51a). \square

Inequality (51b) is much easier to handle with than inequality (45). Choose a function $\psi(u_1) > 0$ at interval $u_1 \in [u_1^-, u_\odot]$ and define another new function $\Delta(u_1, g; \psi)$,

$$\Delta(u_1, g; \psi) = A\psi^2(u_1) + E(u_1)\psi(u_1) - [J(g) + Cu_1(u_1 - 1)]. \quad (52)$$

Proposition 2 *The following relations are equivalent:*

$$a) \quad \Psi_4^+(u_1, g) < \psi(u_1) \quad \Longleftrightarrow \quad b) \quad \Delta(u_1, g; \psi) > 0, \quad (53)$$

$$c) \quad \Psi_4^+(u_1, g) = \psi(u_1) \quad \Longleftrightarrow \quad d) \quad \Delta(u_1, g; \psi) = 0. \quad (54)$$

Proof $a) \Rightarrow b)$. Substituting expression (50) for $\Psi_4^+(u_1, g)$ into (53a) and keeping in mind $A > 0$, we arrive at (53b).

$b) \Rightarrow a)$. Substituting (52) into (53b), and multiplying by $4A$, $A > 0$, and adding $E^2(u_1)$ to the both sides of inequality we get $[2A\psi(u_1) + E(u_1)]^2 > E^2(u_1) + 4A[J(g) + Cu_1(u_1 - 1)]$. The last inequality results in (53a).

$c) \Rightarrow d)$ and $d) \Rightarrow c)$ are similar to each other since the quadratic equation $\Delta(u_1, g; \psi) = 0$ is related to the one of its roots, $\psi(u_1) = \Psi_4^+(u_1, g)$ given in (50). \square

By consequence of (53,54), one more equivalence holds,

$$\Psi_4^+(u_1, g) > \psi(u_1) \iff \Delta(u_1, g; \psi) < 0, \quad (55)$$

Consider two inequalities for u_2 : the 2nd inequality in (35), Lemma 2, and the 2nd inequality in (51), Proposition 1. Find such critical value g_{cr} that for $g > g_{cr}$ both inequalities are satisfied in the range $u_1^- < u_1 \leq u_\odot$, however for any $g < g_{cr}$ there exist always such u_1, u_2 , satisfying (51) but breaking (35). In Lemma 4 we show that such value g_{cr} does exist and it is equal g_s given in (46).

Lemma 4 *Let a symmetric (not CI) numerical semigroup S_5 be given with its Hilbert series $H(S_5; z)$ according to (1). If $u_1 \in [u_1^-, u_\odot]$ then the following hold,*

$$\Psi_4^+(u_1, g) < \Psi_2(u_1), \quad \text{if } g > g_s; \quad \Psi_4^+(u_1^-, g_s) = u_2^*. \quad (56)$$

If $g < g_s$ then there exists such range $u_1 \in [u_1^-, u_1^\bullet)$ where $\Psi_4^+(u_1, g) > \Psi_2(u_1)$.

Proof Substitute $\psi(u_1) = \Psi_2(u_1)$ into (52) and represent $\Delta(u_1, g; \Psi_2)$ as follows,

$$\Delta(u_1, g; \Psi_2) = \Delta(u_1, g_s; \Psi_2) + J(g_s) - J(g), \quad (57)$$

Factorizing a quartic in u_1 polynomial $\Delta(u_1, g_s; \Psi_2)$ and substituting (40) for $J(g)$ into (57), we obtain

$$\begin{aligned} \Delta(u_1, g; \Psi_2) &= \frac{(u_1 - u_1^\ominus)(u_1 - u_1^-)(u_1 - u_1^+)(u_1 - u_1^\oplus)}{32\beta_1(\beta_1 - 1)} + 6\pi_5 \left(\frac{1}{g_s^4} - \frac{1}{g^4} \right), \\ u_1^\ominus &= \frac{\beta_1 - \sqrt{(2\beta_1 - 3)(\beta_1 - 2)}}{2}, \quad u_1^\oplus = \frac{\beta_1 + \sqrt{(2\beta_1 - 3)(\beta_1 - 2)}}{2}, \end{aligned} \quad (58)$$

where u_1^\pm are given in (39). Four roots of polynomial $\Delta(u_1, g_s; \Psi_2)$, and the number u_\odot and the running variable u_1 are arranged as follows,

$$u_1^\ominus < u_1^- \leq u_1 \leq u_\odot < u_1^+ < u_1^\oplus. \quad (59)$$

By virtue of (58) and (59), if $g > g_s$ then $\Delta(u_1, g; \Psi_2) > 0$. Therefore, by equivalence (53) in Proposition 2, we have $\Psi_4^+(u_1, g) < \Psi_2(u_1)$. Moreover, if $g = g_s$, then $\Delta(u_1, g_s; \Psi_2) = 0$ if and only if $u_1 = u_1^-$, i.e., $\Delta(u_1^-, g_s; \Psi_2) = 0$. Thus, by equivalence (54) in Proposition 2, it results in $\Psi_4^+(u_1^-, g) = \Psi_2(u_1^-) = u_2^*$ and (56) is proven.

To prove the 2nd part of Lemma we consider $\Delta(u_1, g; \Psi_2)$ with $g < g_s$ that gives $\Delta(u_1^-, g; \Psi_2) < 0$. By equivalence (55), this leads to $\Psi_4^+(u_1^-, g) > \Psi_2(u_1^-)$. However, due to continuity of the functions

$\Psi_4^+(u_1, g)$ and $\Psi_2(u_1)$ in variable u_1 , the last inequality is also valid in finite vicinity $[u_1^-, u_1^\bullet]$ of u_1^- , where u_1^\bullet is the nearest (from the right) to u_1^- root of quartic equation,

$$(u_1^\bullet - u_1^\ominus)(u_1^\bullet - u_1^-)(u_1^\bullet - u_1^+)(u_1^\bullet - u_1^\oplus) = 192\pi_5\beta_1(\beta_1 - 1) \left(\frac{1}{g^4} - \frac{1}{g_s^4} \right), \quad (60)$$

i.e., if $g < g_s$ then there exists such range $u_1 \in [u_1^-, u_1^\bullet]$ where $\Psi_4^+(u_1, g) > \Psi_2(u_1)$. \square

In Figure 1 we illustrate the behavior of functions, $\Psi_1(u_1)$, $\Psi_2(u_1)$, $\Psi_4^+(u_1, g_s)$ and a distribution of the figurative points (u_1^j, u_2^j) for symmetric (not CI) numerical semigroups A_j^6 , $j \leq 10$, with $\beta_1 = 6$,

$$\begin{aligned} A_1^6 &= \langle 14, 15, 16, 18, 26 \rangle, \quad A_2^6 = \langle 20, 21, 24, 27, 39 \rangle, \quad A_3^6 = \langle 10, 11, 12, 14, 16 \rangle, \quad A_4^6 = \langle 14, 16, 17, 18, 26 \rangle, \\ A_5^6 &= \langle 16, 21, 26, 30, 34 \rangle, \quad A_6^6 = \langle 23, 24, 39, 45, 51 \rangle, \quad A_7^6 = \langle 302, 305, 308, 314, 316 \rangle, \\ A_8^6 &= \langle 302, 308, 314, 315, 316 \rangle, \quad A_9^6 = \langle 453, 462, 469, 471, 474 \rangle, \quad A_{10}^6 = \langle 10, 11, 12, 13, 15 \rangle, \end{aligned}$$

in the $\{u_1, u_2\}$ -plane with different visual resolutions. The coordinates of the points (u_1^j, u_2^j) are given in Table 1, section 6.

Lemma 4 does not state the relation between u_1^\bullet and u_\odot . However, if $g < g_s$ and g and g_s are close enough, then $u_1^\bullet < u_\odot$. Indeed, let $g = g_s(1 - \xi)$, $\xi \ll 1$, and u_1^\bullet be represented as $u_1^\bullet = u_1^- + \xi q$, $q > 0$. Substitute g and u_1^\bullet into (60) and find q in the zero order of ξ ,

$$u_1^\bullet = u_1^- + \frac{8}{\sqrt{\beta_1}} \frac{\beta_1 - 1}{\beta_1 - 3} \xi \quad \rightarrow \quad u_1^\bullet < u_\odot \quad \text{if} \quad \xi \ll 1.$$

Theorem 1 *Let a symmetric (not CI) numerical semigroup S_5 be given with its Hilbert series $H(S_5; z)$ according to (1). Then $g \geq g_s$. The greatest lower bound F_{5s} of the Frobenius number is $F_{5s} = g_s - \sigma_1$.*

Proof In the range $u_1 \in [0, u_1^-]$ the proof is given in Lemma 3. In the range $u_1 \in [u_1^-, u_\odot]$, due to Lemma 4, the following relations hold: $\Psi_4^+(u_1^-, g_s) = u_2^*$ and if $g \geq g_s$ then $\Psi_4^+(u_1, g) < \Psi_2(u_1)$. Combining the last with Proposition 1 we get $u_2 < \Psi_4^+(u_1, g) < \Psi_2(u_1)$.

On the other hand, Lemma 4 states that inequality $g < g_s$ allows to exist such $u_1 \in [u_1^-, u_1^\bullet]$ where $\Psi_4^+(u_1, g) > \Psi_2(u_1)$. In other words, inequality (45) admits such $u_1 \in [u_1^-, u_1^\bullet]$, $u_2 \in [u_2^*, \Psi_4^+(u_1, g)]$ that u_2 may exceed $\Psi_2(u_1)$ and contradict (35) in Lemma 2. Therefore, to exclude those cases, we have to restrict the appropriate values of g by g_s from below that proves Theorem.

Formula for the greatest lower bound F_{5s} of the Frobenius number $F(S_5)$ follows by relationship of the latter with g given in section 1. \square

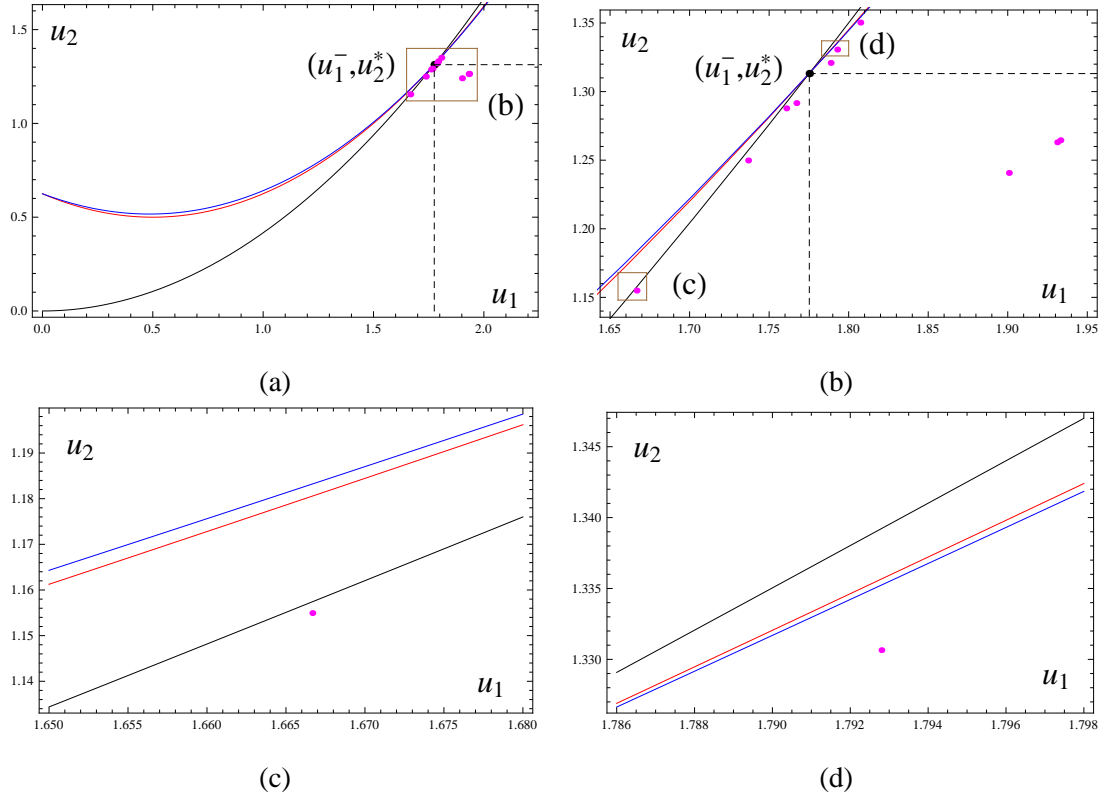


Figure 1: Plots of three functions, $\Psi_1(u_1)$ (black), $\Psi_2(u_1)$ (red), $\Psi_4^+(u_1, g_s)$ (blue), and a distribution of the figurative points (magenta) (u_1^j, u_2^j) for symmetric (not CI) semigroups A_j^6 , $j \leq 10$, with $\beta_1 = 6$ in the $\{u_1, u_2\}$ -plane with different visual resolutions. In a) and b) the brown rectangles, labeled with (b), (c), (d), define the windows enlarged in b), c), d), respectively. In c) and d) the magenta points are related to semigroups A_{10}^6 and A_6^6 , respectively. The special values read: $u_1^- = 1.775$, $u_2^* = 1.3131$.

Compare F_{5s} with two other lower bounds of Frobenius numbers for symmetric CI [6] F_{CI_5} and nonsymmetric [10] F_{NS_5} semigroups generated by five elements,

$$F_{CI_5} = 4\sqrt[4]{\pi_5} - \sigma_1, \quad F_{NS_5} = \sqrt[4]{24\pi_5} - \sigma_1, \quad F_{NS_5} < F_{5s} < F_{CI_5}. \quad (61)$$

5 Symmetric (not CI) semigroups S_5 with W property

Watanabe [14] gave a construction of numerical semigroup S_m generated by m elements starting with semigroup S_{m-1} generated by $m - 1$ elements and proved Lemma.

Lemma 5 [14] Let $S_{m-1} = \langle \delta_1, \dots, \delta_{m-1} \rangle$ and $a > 1$, $d_m > m$, such that $\gcd(a, d_m) = 1$ and $d_m \in S_{m-1}$. If we put $S_m = \langle a\delta_1, \dots, a\delta_{m-1}, d_m \rangle$, which we denote $S_m = \langle d_m, aS_{m-1} \rangle$, then S_m is symmetric iff S_{m-1} is symmetric, and S_m is symmetric CI iff S_{m-1} is symmetric CI.

For our purpose the following obvious Corollary of Lemma 5 is important.

Corollary 1 Let $S_{m-1} = \langle \delta_1, \dots, \delta_{m-1} \rangle$ and $a > 1$, $d_m > m$, such that $\gcd(a, d_m) = 1$ and $d_m \in S_{m-1}$. If we put $S_m = \langle d_m, aS_{m-1} \rangle$, then S_m is symmetric (not CI) iff S_{m-1} is symmetric (not CI).

To utilize this construction we define the following property.

Definition 1 A numerical semigroup S_m has W property if there exists such semigroup S_{m-1} that gives rise to S_m by construction described in Lemma 5 and Corollary 1.

A symbol W stands for Kei-ichi Watanabe. The next Proposition distinguishes the minimal edim of symmetric CI and symmetric (not CI) numerical semigroups with W property.

Proposition 3 A minimal edim of symmetric (not CI) semigroup S_m with W property is $m = 5$.

Proof All numerical semigroups with $\text{edim} = 2$ are symmetric CI, and all symmetric numerical semigroups with $\text{edim} = 3$ are CI [9]. Therefore, according to Definition 1, a minimal edim of symmetric CI semigroups with W property is $\text{edim} = 3$. A minimal edim of numerical semigroups, which are symmetric (not CI), is $\text{edim} = 4$. Therefore, according to Definition 1 and Corollary 1, a minimal edim of symmetric (not CI) semigroups with W property is $\text{edim} = 5$. \square

According to Proposition 3, let us choose a symmetric (not CI) semigroup of $\text{edim} = 5$ with W property and denote it by W_5 in order to distinguish it from symmetric (not CI) semigroups S_5 (irrelevantly to W property), $\{W_5\} \subset \{S_5\}$. A minimal free resolution associated with W_5 was described in [1], section 4, where degrees of all syzygies were also derived ([1], Corollary 12), e.g., its 1st Betti number is $\beta_1 = 6$.

Lemma 6 Let two symmetric (not CI) numerical semigroups $W_5 = \langle aS_4, d_5 \rangle$ and $S_4 = \langle \delta_1, \delta_2, \delta_3, \delta_4 \rangle$ be given and $\gcd(a, d_5) = 1$, $d_5 \in S_4$. Let the greatest lower bound F_{5w} of its Frobenius number $F(W_5)$ be represented as, $F_{5w} = g_w - \left(a \sum_{j=1}^4 \delta_j + d_5\right)$. Then

$$g_w = a \left(\sqrt[3]{25\pi_4(\delta)} + d_5 \right), \quad \pi_4(\delta) = \prod_{j=1}^4 \delta_j. \quad (62)$$

Proof Consider symmetric (not CI) numerical semigroup S_4 generated by four integers, (without W property) and apply the recent result [8] on the greatest lower bound F_{4s} of its Frobenius number $F(S_4)$,

$$F(S_4) \geq F_{4s}, \quad F_{4s} = h_s - \sum_{j=1}^4 \delta_j, \quad h_s = \sqrt[3]{25\pi_4(\delta)}, \quad \sqrt[3]{25} \simeq 2.924. \quad (63)$$

where $\pi_4(\delta)$ is denoted in (62). By relationship [2] between $F(W_5)$ and $F(S_4)$ we have,

$$F(W_5) = aF(S_4) + (a-1)d_5. \quad (64)$$

Represent $F(S_4) = h - \sum_{j=1}^4 \delta_j$ and $F(W_5) = g - a \sum_{j=1}^4 \delta_j - d_5$, substitute them into (64) and get

$$g - a \sum_{j=1}^4 \delta_j - d_5 = ah - a \sum_{j=1}^4 \delta_j + (a-1)d_5 \quad \rightarrow \quad g = a(h + d_5). \quad (65)$$

Thus, comparing the last equality in (65) with the lower bound of h in (63) we arrive at (62). \square

Proposition 4 *Let two symmetric (not CI) numerical semigroups $W_5 = \langle aS_4, d_5 \rangle$ and $S_4 = \langle \delta_1, \delta_2, \delta_3, \delta_4 \rangle$ be given with Hilbert series $H(W_5; z)$ according to (1) and $\gcd(a, d_5) = 1$, $d_5 \in S_4$. Then the following inequality holds,*

$$g_w > g_s. \quad (66)$$

Proof Keeping in mind [1] that $\beta_1 = 6$, determine g_s for W_5 in accordance with (46),

$$g_s = a\bar{\lambda} \sqrt[4]{d_5} \sqrt[4]{\pi_4(\delta)}, \quad \bar{\lambda} = 4\lambda(6) = 4\sqrt[4]{9/10} \simeq 3.896,$$

and calculate a ratio ρ ,

$$\rho = \frac{g_w}{g_s} = \frac{1}{\bar{\lambda}} \frac{\sqrt[3]{25\pi_4(\delta)} + d_5}{\sqrt[4]{d_5} \sqrt[4]{\pi_4(\delta)}} = \frac{1}{\bar{\lambda}} \left(\sqrt[3]{25} \sqrt[4]{\frac{\sqrt[3]{\pi_4(\delta)}}{d_5}} + \sqrt[4]{\frac{d_5^3}{\pi_4(\delta)}} \right). \quad (67)$$

Represent expression in (67) as a function $\rho(\eta)$ in one variable and rewrite it as follows,

$$\rho(\eta) = \frac{1}{\bar{\lambda}} \left(\sqrt[3]{25\eta} + \frac{1}{\eta} \right), \quad \eta = \sqrt[4]{\frac{\pi_4(\delta)}{d_5^3}}. \quad (68)$$

The function $\rho(\eta)$ is positive for $\eta > 0$ and has an absolute minimum

$$\rho(\eta_m) = 1.00713, \quad \eta_m = 1.01943, \quad \pi_4(\delta) \simeq 1.08d_5^3, \quad (69)$$

i.e., we have $\rho(\eta_m) > 1$ that proves Proposition. \square

By definition of the Frobenius number and Proposition 4 we get another inequality, $F_{5w} > F_{5s}$.

6 Numerical experiments with symmetric (not CI) semigroups S_5

In this section we present the numerical results for parameters of symmetric (not CI) semigroups S_5 with different $\beta_1 = 6, 7, 8, 9, 13$. Notations in Tables 1, 2 are defined throughout previous sections.

Table 1. Parameters of symmetric (not CI) semigroups A_j^6 , $j \leq 9$, with W property and A_{10}^6 without it. All semigroups are presented in section 4; their 1st Betti number is $\beta_1 = 6$; $u_1^- = 1.775$, $u_2^* = 1.3131$.

S_5	A_1^6	A_2^6	A_3^6	A_4^6	A_5^6	A_6^6	A_7^6	A_8^6	A_9^6	A_{10}^6
u_1	1.789	1.737	1.761	1.767	1.807	1.793	1.933	1.931	1.901	1.666
u_2	1.321	1.25	1.288	1.292	1.35	1.331	1.265	1.263	1.241	1.155
g	142	228	92	146	218	333	9120	9140	14172	90
g_s	138	222.4	90.85	142.4	212.9	326.6	5046.5	5087.4	8429.4	86.4
g_w	139.4	224.2	91.52	143.4	216.4	330.6	5478.1	5498.1	8709.2	—
η	1.1804	0.9513	1.0599	1.0746	1.3008	1.2151	2.1234	2.0727	1.5377	—

Below we present twelve symmetric (not CI) semigroups S_5 with different $\beta_1 = 7, 8, 9, 13$.

$$\begin{aligned}
\beta_1 = 7, \quad & A_1^7 = \langle 6, 10, 14, 15, 19 \rangle, \quad A_2^7 = \langle 6, 10, 14, 17, 21 \rangle, \quad A_3^7 = \langle 9, 10, 11, 13, 17 \rangle, \\
\beta_1 = 8, \quad & A_1^8 = \langle 6, 10, 14, 19, 23 \rangle, \quad A_2^8 = \langle 8, 10, 13, 14, 19 \rangle, \quad A_3^8 = \langle 8, 9, 12, 13, 19 \rangle, \\
\beta_1 = 9, \quad & A_5^9 = \langle 7, 12, 13, 18, 23 \rangle, \quad A_6^9 = \langle 9, 12, 13, 14, 19 \rangle, \quad A_7^9 = \langle 8, 11, 12, 15, 25 \rangle, \\
\beta_1 = 13, \quad & A_1^{13} = \langle 19, 23, 29, 31, 37 \rangle, \quad A_2^{13} = \langle 19, 27, 28, 31, 32 \rangle, \quad A_3^{13} = \langle 23, 28, 32, 45, 54 \rangle.
\end{aligned}$$

Semigroups $A_j^6, j \leq 9$, with W property are based on symmetric (not CI) semigroups, generated by four integers [8], and found by author. Semigroups $A_1^{13}, A_2^{13}, A_3^{13}$ in Table 2 were studied by Bresinsky [3], the other semigroups of Table 2 and A_6^{10} in Table 1 were generated numerically with help of the package "numericalsgps" by M. Delgado.

Table 2. Parameters of symmetric (not CI) semigroups with $\beta_1 = 7, 8, 9, 13$ without W property.

S_5	A_1^7	A_2^7	A_3^7	A_1^8	A_2^8	A_3^8	A_5^9	A_6^9	A_7^9	A_1^{13}	A_2^{13}	A_3^{13}
u_1	2.172	2.26	2.2	2.616	2.2	2.548	3	3	3	4.75	4.75	4.77
u_1^-	2.177	2.177	2.177	2.586	2.586	2.586	3	3	3	4.697	4.697	4.697
u_2	2.015	2.1	2.065	2.97	2.065	2.93	3.982	3.98	3.97	10.39	10.4	10.45
u_2^*	2.031	2.031	2.031	2.925	2.925	2.925	4	4	4	10.183	10.183	10.183
g	87	93	85	99	89	84	102	96	100	240	236	331
g_s	84.8	89.7	82.9	94.4	87.9	82.4	99.4	94.8	96.2	234.5	233.3	319.5

A comparison of values in pairs, $u_1 - u_1^-$ and $u_2 - u_2^*$, shows that the figurative points (u_1, u_2) for semigroups from Table 2 are distributed in different parts of their existence area given by (34, 35) of Lemma 2.

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